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The Principal Series of $Sp(n, \mathbb{R})$

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Abstract

The principal series of unitary representations of the noncompact symplectic group $Sp(n, \mathbb{R})$ is constructed for all n. The Lie algebra of $Sp(n, \mathbb{R})$ is isomorphic to the algebra of bilinear products of boson operators in n dimensions. The spectrum of the number operator for the principal series representations is shown to be unbounded, both from above and from below.

1. *Introduction*

The real symplectic group $Sp(n) = Sp(n, \mathbb{R})$ is the noncompact simple Lie group given by

$$Sp(n) = \{g \in M_{2n}(\mathbb{R}) \mid g^t J g = J\} \tag{1.1}$$

where $M_{2n}(\mathbb{R})$ denotes the set of $2n \times 2n$ real matrices, elements of which are typically written as

$$g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}, g_i \in M_n(\mathbb{R})$$

and

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in M_{2n}(\mathbb{R})$$

In this article the principal series of unitary representations of $Sp(n)$ is constructed (Theorem 2.3). These representations also define skew-adjoint representations of the real Lie algebra $sp(n)$ of the group $Sp(n)$,

$$sp(n) = \left\{x = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1{}^t \end{pmatrix} \in M_{2n}(\mathbb{R}) \mid x_2{}^t = x_2, x_3{}^t = x_3\right\} \tag{1.2}$$

This Lie algebra is isomorphic to the algebra of bilinear products of boson creation and destruction operators $a_\alpha^\dagger, a_\alpha$ in n dimensions (Lipkin, 1965)

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}, \qquad [a_\alpha, a_\beta] = [a_\alpha^\dagger, a_\beta^\dagger] = 0 \qquad \text{for } 1 \leq \alpha, \beta \leq n \quad (1.3)$$

The explicit isomorphism is given by

$$(a_\alpha^\dagger a_\beta - a_\beta^\dagger a_\alpha) \to (E_{\alpha\beta} - E_{\beta\alpha}) + (E_{\alpha+n,\beta+n} - E_{\beta+n,\alpha+n}) \quad (1.4a)$$

$$(a_\alpha a_\beta - a_\beta^\dagger a_\alpha^\dagger) \to -(E_{\alpha\beta} + E_{\beta\alpha}) + (E_{\alpha+n,\beta+n} + E_{\beta+n,\alpha+n}) \quad (1.4b)$$

$$(i/2)(a_\alpha^\dagger a_\beta + a_\alpha a_\beta^\dagger + a_\alpha a_\beta + a_\alpha^\dagger a_\beta^\dagger) \to -(E_{\alpha+n,\beta} + E_{\beta+n,\alpha}) \quad (1.4c)$$

$$(i/2)(a_\alpha^\dagger a_\beta + a_\alpha a_\beta^\dagger - a_\alpha a_\beta - a_\alpha^\dagger a_\beta^\dagger) \to (E_{\alpha,\beta+n} + E_{\beta,\alpha+n}) \quad (1.4d)$$

where $E_{\alpha\beta} \in M_{2n}(\mathbb{R})$ denotes the matrix whose only nonzero entry at the intersection of the α row and β column equals unity.

We shall show that the principal series of $Sp(n)$ is indexed by pairs (ϵ, ν), where $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ is an n-tuple with integral entries $\epsilon_r = 0$ or 1 and $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ is an n-tuple of real numbers. Moreover, the spectrum of the number operator

$$\sum_{\alpha=1}^{n} (a_\alpha^\dagger a_\alpha + 1/2) \quad (1.5)$$

for the (ϵ, ν) representation is the set of all even (odd) integers if $\Sigma_{r=1}^{n} \epsilon_r$ is even (odd), cf. equation 2.30. As a consequence of this unbounded (from above and below!) spectrum, there is no ground state $|\rangle$ satisfying $a_\alpha a_\beta |\rangle = 0$.

A single representation of $sp(1)$ has previously been obtained with the spectrum of the number operator unbounded from above and below by Biedenharn and Louck (1971). They outline the possible applicability of $Sp(3)$ representations to the problem of extending (dichotomic) conjugation symmetry (s-parity) (Biedenharn, 1969) to a complete classification scheme for rotational bands of $SU(3)$, a subgroup of $Sp(3)$ (Racah, 1964).

However, as an algebraic model in the sense of Rosensteel and Rowe (1975), the fact that the number operator is unbounded from below precludes their application to collective motion problems.

We call to the reader's attention that the Lie algebras $su(1,1)$, $so(2,1)$ and $sl(2)$ are all isomorphic to $sp(1)$. Representations of these Lie algebras have been obtained by many authors (Ui, 1968; Bargmann, 1947; Gelfand and Graev, 1956; Goshen and Lipkin, 1959; Itzykson, 1967).

2. *Principal Series Representations of Sp(n)*

The principal series for $Sp(n)$ is given in Theorem 2.3. As necessary results for the construction of this series, we obtain the Cartan (Proposition 2.1) and Iwasawa (Proposition 2.2 and its corollary) decompositions of $Sp(n)$. As general references in Lie group representation theory, we recommend in order of increasing mathematical complexity Hermann (1966), Mackey (1963), and Warner (1972).

The Killing form on the Lie algebra $sp(n)$, equation (1.2), is given by

$$(x, y) = (2n + 2)\mathrm{Tr}(xy) \tag{2.1}$$

Given this, the Cartan decomposition is immediate.

Proposition 2.1. A Cartan decomposition for $sp(n)$ is given by

$$sp(n) \simeq k \oplus p \qquad \text{(vector space direct sum)}$$

where the Killing form is negative-definite on k,

$$k = \left\{ x = \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix} \mid x_1{}^t = -x_1, x_2{}^t = x_2 \right\}$$

positive-definite on p,

$$p = \left\{ y = \begin{pmatrix} y_1 & y_2 \\ y_2 & -y_1 \end{pmatrix} \mid y_1{}^t = y_1, y_2{}^t = y_2 \right\}$$

and the map

$$k \oplus p \to k \oplus p$$

$$x + y \to x - y, \qquad x \in k, \quad y \in p$$

is an automorphism (Cartan involution) of $sp(n)$.

A maximal abelian subspace a of p is given by the diagonal matrices

$$a = \left\{ H = \sum_{i=1}^{n} \omega_i H_i \mid H_i = E_{ii} - E_{i+n,\, i+n} \right\} \tag{2.2}$$

Let a^* be the real dual of a. If $\alpha \in a^*$, then set

$$g_\alpha = \{X \in sp(n) \mid [X, H] = \alpha(H)X, \quad \text{all } H \in a\}$$

If $g_\alpha \neq \{0\}$ and $\alpha \neq 0$, then α is called a restricted root. Now for the simple Lie algebra, $sp(n)$, the dimension of g_α for each restricted root α is unity. In the following list is given an element $e_\alpha \in g_\alpha$ for each restricted root α:

$$\begin{aligned} e_{\omega_j - \omega_i} &= E_{ij} - E_{j+n,\, i+n}, && i \neq j \\ e_{-\omega_i - \omega_j} &= E_{i,\, j+n} + E_{j,\, i+n}, && i < j \\ e_{\omega_i + \omega_j} &= E_{i+n,\, j} + E_{j+n,\, i}, && i < j \\ e_{-2\omega_i} &= E_{i,\, i+n} \\ e_{2\omega_i} &= E_{i+n,\, i} \end{aligned} \tag{2.3}$$

where $\omega_j - \omega_i$ denotes the root

$$H = \sum_{i=1}^{n} \omega_i H_i \to \omega_j - \omega_i, \text{ etc.}$$

Let a' denote the set of $H \in a$ with $\alpha(H) \neq 0$ for every restricted root α. A Weyl chamber is a connected component in a'. Fix the Weyl chamber a^+,

$$a^+ = \left\{ H = \sum_{i=1}^{n} \omega_i H_i \,|\, \omega_1 > \omega_2 > \cdots > \omega_n > 0 \right\} \tag{2.4}$$

Let Σ^+ denote the set of positive restricted roots, i.e., those restricted roots α such that $\alpha(H) > 0$ for all $H \in a^+$. Put

$$n = \bigoplus_{\alpha \in \Sigma^+} g_\alpha = \left(\bigoplus_{i<j} \mathbb{R} e_{\omega_j - \omega_i} \right) \oplus \left(\bigoplus_{i<j} \mathbb{R} e_{\omega_j + \omega_i} \right) \oplus \left(\bigoplus_{i} \mathbb{R} e_{2\omega_i} \right) \tag{2.5}$$

where $\mathbb{R}e_\alpha$ denotes the one-dimensional subspace spanned by e_α. Then n is a nilpotent Lie algebra. We have now obtained the Iwasawa decomposition.

Proposition 2.2. The Iwasawa decomposition of $sp(n)$ is given by

$$sp(n) \simeq k \oplus a \oplus n \qquad \text{(vector space direct sum)}$$

If K, A, N are the analytic subgroups in $Sp(n)$ corresponding to k, a, n, respectively, then the map

$$K \times A \times N \to Sp(n)$$

$$(x, h, z) \to x \cdot h \cdot z, \qquad x \in K, \quad h \in A, \quad z \in N$$

is an analytic diffeomorphism onto $Sp(n)$.

Since $k = so(2n) \cap sp(n)$, then $K = SO(2n) \cap Sp(n)$. Moreover, K may be identified with $U(n)$ via the isomorphism (Helgason, 1962)

$$U(n) \to K \tag{2.6}$$

$$U + iV \to \begin{pmatrix} U & V \\ -V & U \end{pmatrix}$$

for $U + iV \in U(n)$ with U, V real $n \times n$ matrices. One readily computes that $A = \exp(a)$ is given by

$$A = \{h = \text{diag}(h_1, h_2, \ldots, h_n, h_1^{-1}, h_2^{-1}, \ldots, h_n^{-1}), \quad h_i > 0\} \tag{2.7}$$

The Iwasawa subgroup $Z = A \cdot N$ is shown to be

$$Z = \left\{ z = \begin{pmatrix} z_1 & 0 \\ z_3 & (z_1{}^t)^{-1} \end{pmatrix} \in Sp(n) \right.$$

where

$$z_1 = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ 0 & z_{22} & z_{23} \\ 0 & 0 & z_{33} \end{pmatrix}, \qquad z_{11}, z_{22}, z_{33} > 0 \tag{2.8}$$

A corollary to the above proposition is evident.

Corollary. The following map is an analytic diffeomorphism onto $Sp(n)$:

$$K \times Z \to Sp(n)$$

$$(x, z) \to x \cdot z, \qquad x \in K, \quad z \in Z$$

where $K \simeq U(n)$ via equation (2.6) and the Iwasawa subgroup Z is given in equation (2.8).

If $\rho \in a^*$ is defined by

$$\rho(H) = \tfrac{1}{2}\mathrm{Tr}(adH\,|_n) \qquad \text{for } H \in a \tag{2.9}$$

then

$$\rho(H) = \omega_1 + 2\omega_2 + 3\omega_3 + \cdots + n\omega_n \qquad \text{for } H = \sum_{i=1}^{n} \omega_i H_i \in a \tag{2.10}$$

In general, an element $\nu \in a^*$ is specified by an n-tuple $(\nu_1, \nu_2, \ldots, \nu_n)$ of real numbers,

$$\nu(H) = \sum_{i=1}^{n} \nu_i \omega_i \quad \text{for } H = \sum_{i=1}^{n} \omega_i H_i \in a \tag{2.11}$$

The centralizer of A in K is given by

$$M = \{m = \mathrm{diag}(m_1, m_2, \ldots, m_n, m_1, m_2, \ldots, m_n), \qquad m_i = \pm 1\} \tag{2.12}$$

Via the isomorphism of equation (2.6), M may be identified with the subgroup of diagonal matrices in $U(n)$ with entries $\pm$ 1.

The principal series for $Sp(n)$ is then given as follows (Wallach, 1971): For each pair (ξ, ν), where ξ is a unitary character of M and ν is an element of a^*, a unitary representation $U^{(\xi, \nu)}$ with carrier space $F^{(\xi, \nu)}$ is given. The carrier space

$$F^{(\xi, \nu)} = \{f\colon Sp(n) \to \mathbb{C}|$$

$$\text{(i) } f(gmhz) = e^{-(\rho + i\nu)(\log h)} \xi(m)^{-1} f(g)$$

$$\text{for } g \in Sp(n), \quad m \in M, \quad h \in A, \quad z \in N$$

(Here log: $A \to a$ is the inverse to exp: $a \to A$.)

$$\text{(ii) } \int_{K/M} d\mu(x)\, |f(x)|^2 < \infty\} \tag{2.13}$$

where $d\mu(x)$ is the K-invariant measure on K/M. $F^{(\xi, \nu)}$ is a Hilbert space with the inner product

$$(f_1, f_2) = \int_{K/M} d\mu(x) f_1(x)^* f_2(x) \tag{2.14}$$

The action of the unitary representation $U^{(\xi,\nu)}$ on $F^{(\xi,\nu)}$ is given by

$$(U_{g_0}^{(\xi,\nu)}f)(g) = f(g_0^{-1}g) \qquad \text{for } g_0, g \in Sp(n), \quad f \in F^{(\xi,\nu)} \tag{2.15}$$

A simpler manifestation of the principal series, unitarily equivalent to the above, may be given. If $f \in F^{(\xi,\nu)}$, then set

$$\begin{aligned} &\phi : K \to \mathbb{C} \\ &\phi(x) = f(x), \qquad x \in K \end{aligned} \tag{2.16}$$

Note that

$$\phi(xm) = \xi(m)^{-1}\phi(x), \qquad x \in K, \quad m \in M \tag{2.17}$$

Moreover,

$$f(g) = f(x \cdot z) = e^{-(\rho + i\nu)(\log h)}\phi(x) \tag{2.18}$$

where

$$g = x \cdot z \in K \cdot Z = x \cdot h \cdot z \in K \cdot A \cdot N$$

On the other hand, every $f(g)$ of the form (2.18) where $\phi(x)$ satisfies (2.17) satisfies condition (i) in the definition of $F^{(\xi,\nu)}$. We thus obtain the compact realization of the principal series.

Theorem 2.3. The unitary principal series of $Sp(n)$ is indexed by pairs (ξ, ν), where ξ is a unitary character of M and $\nu \in a^*$. The carrier space Ω^ξ of the (ξ, ν) representation $\pi^{(\xi,\nu)}$ is independent of ν,

$$\begin{aligned} \Omega^\xi = \{\phi : K \to \mathbb{C} \mid \\ \text{(i)} \quad & \phi(xm) = \xi(m)^{-1}\phi(x) \\ & \qquad x \in K, \quad m \in M \\ \text{(ii)} \quad & \int_{K/M} d\mu(x)\, |\phi(x)|^2 < \infty\} \end{aligned}$$

Ω^ξ is a Hilbert space with the inner product

$$(\phi_1, \phi_2) = \int_{K/M} d\mu(x)\phi_1(x)^*\phi_2(x)$$

The action of the unitary representation $\pi^{(\xi,\nu)}$ on the space Ω^ξ is

$$\begin{aligned} (\pi_{g_0}^{(\xi,\nu)}\phi)(x) = \zeta_{11}(g_0^{-1}, x)^{-i\nu_1 - 1}\zeta_{22}(g_0^{-1}, x)^{-i\nu_2 - 2} \cdots \\ \zeta_{nn}(g_0^{-1}, x)^{-i\nu_n - n} \cdot \phi\,[\chi(g_0^{-1}, x)] \end{aligned}$$

where

$$g_0 \in Sp(n), \quad \phi \in \Omega^\xi, \; x \in K$$

and

$$g_0^{-1} \cdot x = \chi(g_0^{-1}, x) \cdot \zeta(g_0^{-1}, x)$$

with

$$\chi(g_0^{-1}, x) \in K, \quad \zeta(g_0^{-1}, x) \in Z$$

The Weyl group W is isomorphic to a semidirect product $[Z_2{}^n] S_n$, where the normal subgroup $Z_2{}^n$ is the direct product of n-copies of Z_2, the multiplicative group with elements ± 1, S_n is the permutation group on n symbols, and the action of S_n on $Z_2{}^n$ is the obvious one. The Weyl group acts on the space of pairs (ξ, ν) by

$$(\xi, \nu) \to (\xi^s, \nu^s) \qquad \text{for } s = (\lambda, p) \in [Z_2{}^n] S_n$$

where for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in Z_2{}^n$ and $p \in S_n$,

$$\xi^s = (\epsilon_{p(1)}, \epsilon_{p(2)}, \ldots, \epsilon_{p(n)}) \qquad \text{for } \xi = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$$

and

$$\nu^s = (\lambda_1 \nu_{p(1)}, \lambda_2 \nu_{p(2)}, \ldots, \lambda_n \nu_{p(n)}) \qquad \text{for } \nu = (\nu_1, \nu_2, \ldots, \nu_n)$$

Then $\pi^{(\xi, \nu)}$ is equivalent to $\pi^{(\xi^s, \nu^s)}$ for all $s \in W$ (Bruhat, 1956). Moreover, $\pi^{(\xi, \nu)}$ is irreducible if (ξ, ν) is inequivalent to (ξ^s, ν^s) for all $s \in W$, $s \neq$ identity (Bruhat, 1965). Hence, $\pi^{(\xi, \nu)}$ is irreducible for almost every pair (ξ, ν).

Action of the Subgroup $U(n)$. The action of $U(n)$ is independent of ν:

$$(\pi_{g_0}^{(\xi, \nu)} \phi)(x) = \phi(g_0^{-1} x) \tag{2.19}$$

for $g_0 \in K$ and $\phi \in \Omega^\xi$, $x \in K$. With the identification of K with $U(n)$ given by equation (2.6), the above action is unitarily equivalent to the left regular representation of $U(n)$ on $L^2 [U(n)]^\xi$, the subspace of square-integrable complex-valued functions on $U(n)$ satisfying

$$\phi(xm) = \xi(m)^{-1} \phi(x) \tag{2.20}$$

for $x \in U(n)$, $m \in M$, $\phi \in L^2 [U(n)]^\xi$.

As guaranteed by the Peter–Weyl theorem, the matrix elements of the inequivalent irreducible unitary representations of $U(n)$ form a basis for the space of square-integrable complex-valued functions on $U(n)$. The inequivalent irreducible unitary representations of $U(n)$ (Gelfand and Zetlin, 1950; Hamermesh, 1962) are indexed by n-tuples of integers $[i] = [i_1 i_2 \cdots i_n]$ with $i_1 \geq i_2 \geq \cdots \geq i_n$; the matrix elements $\mathscr{D}_{qp}^{[i]}(x)$ are indexed in the Gelfand basis by triangular arrays:

$$|p\rangle = \left| \begin{matrix} p_{11} p_{12} p_{13} - - - - p_{1n} \\ p_{22} p_{23} \quad p_{2n} \\ \ddots \quad \cdot^{\cdot^{\cdot}} \\ p_{nn} \end{matrix} \right\rangle \tag{2.21}$$

where the entries $p_{\alpha\beta}$ are integers satisfying

$$i_1 = p_{11}, \qquad i_2 = p_{12}, \ldots, \qquad i_n = p_{1n} \tag{2.22}$$

and the "betweenness" condition,

$$p_{r-1,\alpha-1} \leq p_{r\alpha} \leq p_{r-1,\alpha}, \qquad r = 2, 3, \ldots, n, \qquad \alpha = r, r+1, \ldots, n \tag{2.23}$$

Thus a spanning set for $L^2\,[U(n)]^\xi$ satisfying equation (2.20) is given by

$$\phi_{qp}^{[i]}(x)^\xi = \sum_{m \in M} \xi(m^{-1})^{-1} \mathscr{D}_{qp}^{[i]}(xm) \tag{2.24}$$

A unitary character ξ of M is given by an n-tuple $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ with $\epsilon_j = 0, 1$,

$$\xi(m) = m_1^{\epsilon_1} m_2^{\epsilon_2} \ldots m_n^{\epsilon_n} \tag{2.25}$$

for $m = \mathrm{diag}(m_1, m_2, \ldots, m_n) \in M$. Moreover, we have

$$m\,|p\rangle = m_1^{-w_1} m_2^{-w_2} \cdots m_n^{-w_n}\,|p\rangle \tag{2.26}$$

for $m = \mathrm{diag}(m_1, m_2, \ldots, m_n) \in M$, where the weights are given by

$$\begin{aligned} w_1 &= p_{nn} \\ w_2 &= (p_{n-1,n-1} + p_{n-1,n}) - p_{nn} \\ &\;\vdots \\ w_n &= (p_{11} + p_{12} + \cdots + p_{1n}) - (p_{22} + \cdots + p_{2n}) \end{aligned} \tag{2.27}$$

i.e., w_r is the difference between the sum of the entries of the $(n-r+1)$ row and the sum of the entries of the $(n-r+2)$ row for $r = 2, 3, \ldots, n$ and $w_1 = p_{nn}$. We may thus compute

$$\phi_{qp}^{[i]}(x)^\xi = \prod_{r=1}^{n} [1 + (-1)^{\epsilon_r - w_r}] \cdot \mathscr{D}_{qp}^{[i]}(x) \tag{2.28}$$

Hence, a basis for $L^2\,[U(n)]^\xi$ is given by the set of all $\mathscr{D}_{qp}^{[i]}(x)$, where the indices $[i]$ and p are restricted by the condition

$$\prod_{r=1}^{n} [1 + (-1)^{\epsilon_r - w_r}] \neq 0 \tag{2.29}$$

i.e., ϵ_r even (odd) requires that w_r be even (odd) for all $r = 1, 2, \ldots, n$.

Now the vector $\mathscr{D}_{qp}^{[i]} \in L^2\,[U(n)]^\xi$ is an eigenstate of the number operator belonging to the eigenvalue $\Sigma_{\alpha=1}^n i_\alpha$:

$$\sum_{\alpha=1}^{n} (a_\alpha^\dagger a_\alpha + \tfrac{1}{2})\, \mathscr{D}_{qp}^{[i]} = \left(\sum_{\alpha=1}^{n} i_\alpha \right) \mathscr{D}_{qp}^{[i]} \tag{2.30}$$

Hence, the spectrum of the number operator on $\mathcal{L}^2 [U(n)]^{\xi}$ is the set of all even (odd) integers according to whether $\Sigma_{r=1}^{n} \epsilon_r$ is even (odd).

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